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1991 J. Phys. A: Math. Gen. 24 4981

(http://iopscience.iop.org/0305-4470/24/21/012)

J. Phys. A: Math. Gen. 24 (1991) 4981–4988. Printed in the UK

General potentials described by so(2, 1) dynamical algebra in parabolic coordinate systems

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Received 2 April 1991, in final form 18 June 1991

Abstract. We propose general three-dimensional potentials in rotational and cylindrical parabolic coordinates which are generated by direct products of the SO(2, 1) dynamical group. Then we construct their Green functions algebraically and find their spectra. Particular cases of these potentials which appear in the literature are also briefly discussed.

1. Introduction

The search for soluble potentials in quantum mechanics has attracted great interest. A general discussion of all three-dimensional separable potentials was established for the Schrödinger equation by Eisenhart [1] long ago. Recently, Grosche [2] discussed this technique in the context of path integrals.

Symmetries are frequently invoked to explain well-known solutions and perhaps to generate new ones. Winternitz *et al* [3] constructed arbitrary potentials exhibiting a dynamical group in two dimensions and Makarov *et al* [4] searched for three-dimensional systems with various integrals of motion. This problem was also studied at the classical level by Evans [5]. The relation between the accidental degeneracy and a symmetry Lie algebra was also recently discussed by Moshinsky *et al* [6].

In this work we search for general three-dimensional potentials in parabolic coordinates. This particular choice is due to the fact that they can describe interesting physical systems such as the Coulomb, Hartmann [7] and the Coulomb plus an Aharonov–Bohm [8] potentials among others [9]. The so(2, 1) algebra is known as the spectrum-generating algebra of the one-dimensional harmonic oscillator, Coulomb and Morse potentials [10, 11] as well as some other two- and three-dimensional problems [9, 12–16].

The aim of this work is to give an insight towards the construction of exactly solvable three-dimensional potentials related to a specific dynamical algebra. We briefly review in section 2 how so(2, 1) Lie algebra can generate Green functions of the Schrödinger equation. In section 3 we construct a general three-dimensional potential in parabolic rotational coordinates related to so(2, 1) Lie algebra and obtain algebraically the Green function and calculate the spectrum of the Hamiltonian and the corresponding wavefunctions. In section 4 we briefly discuss a general potential in parabolic cylindrical coordinates related to the same algebra and leave the conclusions to section 5.

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0305-4470/91/214981+08$03.50

2. Green function and so(2, 1) Lie algebra

Let us briefly review in this section the basic features of so(2, 1) Lie algebra and its use in constructing Green functions. The algebra is defined by the commutation relations

$$[\boldsymbol{T}_1, \boldsymbol{T}_2] = -\mathrm{i}\boldsymbol{T}_1 \qquad [\boldsymbol{T}_2, \boldsymbol{T}_3] = -\mathrm{i}\boldsymbol{T}_3 \qquad [\boldsymbol{T}_1, \boldsymbol{T}_3] = -\mathrm{i}\boldsymbol{T}_2 \tag{2.1}$$

and one may construct a faithful representation of generators [16, 17]

$$\boldsymbol{T}_1(x) = \alpha_2 x^{2-j}\frac{\partial^2}{\partial x^2} + \alpha_1 x^{1-j}\frac{\partial}{\partial x} + \alpha_0 x^{-j} \tag{2.2a}$$

$$\boldsymbol{T}_2(x) = -\frac{\mathrm{i}}{j}x\frac{\partial}{\partial x} - \mathrm{i}\beta \tag{2.2b}$$

$$\boldsymbol{T}_3(x) = \lambda x^j \tag{2.2c}$$

where the parameters β and λ are restricted by

$$\beta = \frac{1}{2j}\left(\frac{\alpha_1}{\alpha_2} - 1\right) + \frac{1}{2} \qquad \lambda = -\frac{1}{2\alpha_2 j^2}. \tag{2.3}$$

The Green functions $G_E(\boldsymbol{r}, \boldsymbol{r}')$ satisfies

$$(H - E)G_E(\boldsymbol{r}, \boldsymbol{r}') = \delta^3(\boldsymbol{r} - \boldsymbol{r}') \tag{2.4}$$

where H is the Hamiltonian of a three-dimensional system:

$$H = -\frac{\hbar^2}{2M}\nabla^2 + V(\boldsymbol{r}) \tag{2.5}$$

and E is the energy eigenvalue. We use the Schwinger representation [18] for the Green function:

$$G_E(\boldsymbol{r}, \boldsymbol{r}') = \lim_{\varepsilon\to 0}\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}s\, \exp\left(-\frac{\mathrm{i}s}{\hbar}(H - E - \mathrm{i}\varepsilon)\right)\delta^3(\boldsymbol{r} - \boldsymbol{r}'). \tag{2.6}$$

Writing the resolvent operator $\Lambda(x_i)$ for the effective one-dimensional Hamiltonian H_i with eigenvalue E_i,

$$\Lambda(x_i) = x_i^{2-j}(H_i - E_i) = g_0 + g_1\boldsymbol{T}_1 + g_3\boldsymbol{T}_3 \qquad (i = 1, 2, 3) \tag{2.7}$$

and applying the Milstein and Strakhovenko method [19], with the generators $\boldsymbol{T}_l$ $(l = 1, 2, 3)$ (equations (2.2a)–(2.2c)), it is possible to show that [16, 20]

$$\begin{aligned}
&\exp\left(-\frac{\mathrm{i}s}{\hbar}(g_0 + g_1\boldsymbol{T}_1 + g_3\boldsymbol{T}_3)\right)\delta(x_i - x_i')\\
&\quad = -j\,\mathrm{e}^{2\pi\mathrm{i}\nu}\sigma^{\nu+1}(x_i'x_i)^{(j\nu+1)/2}\exp\left(-\frac{\sigma}{2}(x_i^j + x_i'^j)\right)\sum_{n=0}^{\infty}\frac{n!}{\Gamma(n+\nu+1)}\\
&\quad\quad \times L_n^\nu(\sigma x_i^j)L_n^\nu(\sigma x_i'^j)\exp\left(-\mathrm{i}\frac{s}{\hbar}[g_0 + k(\nu + 1 + 2n)]\right)
\end{aligned} \tag{2.8}$$

where

$$\sigma = 4\lambda k/g_1 \qquad k = (g_1 g_3/2)^{1/2} \qquad \nu = \pm\frac{1}{j}\left[\left(\frac{\alpha_2}{\alpha_1} - 1\right)^2 - 4\frac{\alpha_0}{\alpha_2}\right]^{1/2} \tag{2.9}$$

and $L_n^\nu(x)$ are the Laguerre polynomials [21]. Then equation (2.8) can be used for each coordinate x_i ($i=1,2,3$) of a three-dimensional system if the effective Hamiltonian H_i or, equivalently, the resolvent operator $\Lambda(x_i)$ (equation (2.7)) can be written as a linear combination on the generators $T_l(x_i)$ of the so(2, 1) Lie algebra. Once equation (2.8) is substituted in equation (2.6), the integration over the parameter s can be carried out and the poles of the Green function are given by [16]

$$g_0+k(\nu+1+2n)=0 \qquad (n=0,1,2,\ldots) \tag{2.10}$$

which give the eigenvalues of the effective Hamiltonian H_i, since the parameters g_0, g_3 or α_0 are related to E_i.

3. Parabolic rotational coordinates

We want to find a general three-dimensional potential $V(\boldsymbol{r})$ in parabolic rotational coordinates ξ, η, φ defined by

$$x=\xi\eta\cos\varphi \qquad y=\xi\eta\sin\varphi \qquad z=(\eta^2-\xi^2)/2 \tag{3.1}$$

where $0\leqslant\xi<\infty$, $0\leqslant\eta<\infty$ and $0\leqslant\varphi\leqslant2\pi$, which possesses the so(2, 1) dynamical Lie algebra, in such a way that its Hamiltonian can be expressed as a resolvent operator (equation (2.7)). As the Laplacian in this coordinate system is given by

$$\nabla^2=\frac{1}{\xi^2+\eta^2}\left(\frac{\partial^2}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial}{\partial\xi}+\frac{\partial^2}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial}{\partial\eta}\right)+\frac{1}{\xi^2\eta^2}\frac{\partial^2}{\partial\varphi^2} \tag{3.2}$$

it cannot be written directly as a linear combination of the generators T_l ($l=1,2,3$), hence we propose a potential $V(\xi,\eta)$, to be of the form

$$V(\xi,\eta)=\frac{1}{\xi^2+\eta^2}\left(a_1\xi^2+\frac{b_1}{\xi^2}+a_2\eta^2+\frac{b_2}{\eta^2}+c\right). \tag{3.3}$$

This is not only a matter of choice but an imposition that comes from the fact that we want the relation (2.7) to hold or, in other words, that the potential $V(\xi,\eta)$ can be described by so(2, 1) dynamical algebra. If one compares the Laplacian (equation (3.2)) and the potential (equation (3.3)) it is clear that the effective potentials associated with the coordinates ξ and η, apart from the factor $(\xi^2+\eta^2)^{-1}$, will be of the quadratic type $(x_i^2+x_i^{-2})$. Note that the operators of equations (2.2a)-(2.2c) permit, in principle, any power potential (with $j\neq0$), but unfortunately this arbitrariness is restricted by the presence of the energy term in equation (2.7), which in general also has x_i dependence, which must be described by the generators $T_l(x_i)$ too. In the case of Laplacian (3.2), the factor $(\xi^2+\eta^2)^{-1}$ implies that $j=2$, in equation (2.2a)-(2.2c).

In general we see that each type of coordinate system imposes its own features on the so(2, 1) generators. As an example of a different behaviour, in spherical polar coordinates the parameter j can also be equal to one, which corresponds to the Coulomb potential [16, 17].

Before we solve equation (2.4) with the potential (3.3), let us write the latter in spherical polar coordinates r, θ, φ:

$$V(\boldsymbol{r})=\frac{c}{2r}+\frac{b_1+b_2+(b_1-b_2)\cos\theta}{2r^2\sin^2\theta}+\tfrac{1}{2}(a_1+a_2)+\tfrac{1}{2}(a_1-a_2)\cos\theta. \tag{3.4}$$

This potential appears in the general classification of potentials possessing high symmetry [3, 4] and also in the work of Kibler and Winternitz [9] but, as far as we know, its solution has not been published anywhere. This general potential can be reduced, for example, to the Hartmann potential [7] when $a_1 = a_2$ and $b_1 = b_2$. This is also the case of a Coulomb plus an Aharonov-Bohn potential [9]. When all the coupling constants, except c, vanish it reduces to a Coulomb potential.

As the potential (3.3) is φ independent let us use the operator identity

$$A^{-1}B^{-1} = (BA)^{-1} \tag{3.5}$$

in the Schwinger representation (equation (2.6)) to put a factor $(\xi\eta)^2$ in the exponential and rewrite the Green function as

$$G_E(\boldsymbol{r}, \boldsymbol{r}') = \lim_{\varepsilon\to 0} \frac{\mathrm{i}}{\hbar} \int_0^\infty \mathrm{d}s \exp\left(-\mathrm{i}\frac{s}{\hbar}[\xi^2\eta^2(H-E) - \mathrm{i}\varepsilon]\right) \xi^2\eta^2\delta^3(\boldsymbol{r}-\boldsymbol{r}'). \tag{3.6}$$

Now, using equations (2.5), (3.2) and (3.3) we can write

$$\xi^2\eta^2(H-E) = \Lambda(\varphi) + \frac{\xi^2\eta^2}{\xi^2+\eta^2}\Lambda(\xi, \eta) \tag{3.7}$$

where

$$\Lambda(\varphi) = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial\varphi^2} \tag{3.8}$$

and

$$\Lambda(\xi, \eta) = -\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial}{\partial\eta}\right) + (a_1 - E)\xi^2 + \frac{b_1}{\xi^2} + (a_2 - E)\eta^2 + \frac{b_2}{\eta^2} + c \tag{3.9}$$

so equation (3.6) is rewritten as

$$G_E(\boldsymbol{r}, \boldsymbol{r}') = \lim_{\varepsilon\to 0} \frac{\mathrm{i}}{\hbar} \int_0^\infty \mathrm{d}s \exp\left(-\frac{\mathrm{i}}{\hbar} s \frac{\xi^2\eta^2}{\xi^2+\eta^2}\Lambda(\xi, \eta)\right) \frac{\xi\eta\delta(\xi-\xi')\delta(\eta-\eta')}{\xi^2+\eta^2}$$
$$\times \exp\left(-\frac{\mathrm{i}}{\hbar} s\Lambda(\varphi)\right)\delta(\varphi-\varphi') \tag{3.10}$$

where we have used

$$\delta^3(\boldsymbol{r}-\boldsymbol{r}') = \frac{\delta(\xi-\xi')\delta(\eta-\eta')\delta(\varphi-\varphi')}{\xi\eta(\xi^2+\eta^2)} \tag{3.11}$$

and that fact that the operator $\Lambda(\varphi)$ commutes with $[\xi^2\eta^2/(\xi^2+\eta^2)]\Lambda(\xi, \eta)$. Since the operator $\Lambda(\varphi)$ comprises only a second-order derivative and the delta function can be expanded as

$$\delta(\varphi-\varphi') = \sum_{m=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}m(\varphi-\varphi')} \tag{3.12}$$

we find that

$$\exp\left(-\frac{\mathrm{i}}{\hbar} s\Lambda(\varphi)\right)\delta(\varphi-\varphi') = \sum_{m=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}m(\varphi-\varphi')} \exp\left(\mathrm{i}s\frac{\hbar}{2M}m^2\right). \tag{3.13}$$

Substituting equation (3.13) in equation (3.10) and applying again the operator identity (3.5) to introduce a factor $(\xi^2+\eta^2)/(\xi\eta)^2$ in the exponential in order to separate the ξ and η parts we find

$$G_E(\boldsymbol{r},\boldsymbol{r}')=\sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\varphi-\varphi')}\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}s\,\exp\left(-\mathrm{i}\frac{s}{\hbar}[\Lambda_m(\xi)+\Lambda_m(\eta)]\right)\times\frac{\delta(\xi-\xi')\delta(\eta-\eta')}{\xi\eta} \tag{3.14}$$

where

$$\Lambda_m(\xi)=-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial}{\partial\xi}\right)+(a_1-E)\xi^2+\xi^{-2}\left(b_1-\frac{\hbar^2}{2M}m^2\right)+\frac{c}{2} \tag{3.15}$$

and

$$\Lambda_m(\eta)=-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial}{\partial\eta}\right)+(a_2-E)\eta^2+\eta^{-2}\left(b_2-\frac{\hbar^2}{2M}m^2\right)+\frac{c}{2}. \tag{3.16}$$

Writing equation (2.8) for $\Lambda_m(\xi)$ we have

$$\exp\left(-\mathrm{i}\frac{s}{\hbar}\Lambda_m(\xi)\right)\frac{\delta(\xi-\xi')}{\xi}=\sum_{n_1=0}^{\infty}P_{Emn_1}(\xi,\xi')\exp\left[-\mathrm{i}\frac{s}{\hbar}\left(\frac{c}{2}+k_1(\nu_{1m}+1+2n_1)\right)\right] \tag{3.17}$$

where

$$P_{Emn_1}(\xi,\xi')=-2\exp\left(2\mathrm{i}\pi\nu_{1m}\right)\sigma_1(\sigma_1\xi'\xi)^{\nu_{1m}}\exp\left(-\frac{\sigma_1}{2}(\xi^2+\xi'^2)\right)\times\frac{n_1!\,L_{n_1}^{\nu_{1m}}(\sigma_1\xi^2)L_{n_1}^{\nu_{1m}}(\sigma_1\xi'^2)}{\Gamma(n_1+\nu_{1m}+1)} \tag{3.18}$$

and

$$\begin{gathered}\alpha_2=\alpha_1=-1 \qquad \alpha_0=\frac{2M}{\hbar^2}b_1-m^2 \qquad \lambda=\tfrac{1}{8} \qquad g_0=\frac{c}{2}\\ g_1=\frac{\hbar^2}{2M} \qquad g_3=8(a_1-E) \qquad k_1=\hbar\left(\frac{2(a_1-E)}{M}\right)^{1/2}\\ \nu_{1m}=\left(\frac{2M}{\hbar^2}b_1-m^2\right)^{1/2} \qquad \sigma_1=\frac{[2M(a_1-E)]^{1/2}}{\hbar}.\end{gathered} \tag{3.19}$$

Note that the operators $\Lambda_m(\xi)$ and $\Lambda_m(\eta)$ are completely symmetric, so one can easily translate equations (3.17)-(3.19) for $\Lambda_m(\eta)$. Finally, we find the Green function for the potential $V(\xi,\eta)$:

$$G_E(\boldsymbol{r},\boldsymbol{r}')=\sum_{m=-\infty}^{\infty}\sum_{n_1,n_2=0}^{\infty}\mathrm{e}^{\mathrm{i}m(\varphi-\varphi')}P_{Emn_1}(\xi,\xi')P_{Emn_2}(\eta,\eta')\times\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}s\left(-\mathrm{i}\frac{s}{\hbar}[c+k_1(\nu_{1m}+1+2n_1)+k_2(\nu_{2m}+1+2n_2)]\right). \tag{3.20}$$

This solution for the potential $V(\xi,\eta)$ (equation (3.3)) means that this potential is described by a direct product SO(2, 1)⊗SO(2, 1) dynamical group. Performing the integration in the variable s we find the poles of this Green function which constitutes the spectrum of the problem:

$$c+k_1(\nu_{1m}+1+2n_1)+k_2(\nu_{2m}+1+2n_2)=0. \tag{3.21}$$

The wavefunctions $\Psi_{mn_1n_2}(\xi,\eta,\varphi)$ can be obtained from the residues of the Green function at the poles:

$$\begin{aligned}\Psi_{mn_1n_2}(\xi,\eta,\varphi)\\ &=-2\,\mathrm{e}^{im\varphi}\exp[\mathrm{i}\pi(\nu_{1m}+\nu_{2m})]\\ &\quad\times[\sigma_1\sigma_2(\sigma_1\xi^2)^{\nu_{1m}}(\sigma_2\eta^2)^{\nu_{2m}}]^{1/2}\exp(-\tfrac{1}{2}(\sigma_1\xi^2+\sigma_2\eta^2))\\ &\quad\times\left(\frac{n_1!\,n_2!}{\Gamma(n_1+\nu_{1m}+1)\Gamma(n_2+\nu_{2m}+1)}\right)^{1/2}L_{n_1}^{\nu_{1m}}(\sigma_1\xi^2)L_{n_2}^{\nu_{2m}}(\sigma_2\eta^2).\end{aligned} \tag{3.22}$$

Substituting the values of k_1 and k_2 in equation (3.21) we find

$$\begin{aligned}E_{mn_1n_2}=\Bigg\{&(a_1A_{mn_1}^2-a_2B_{mn_2}^2)(A_{mn_1}^2-B_{mn_2}^2)+\frac{Mc^2}{2\hbar^2}(A_{mn_1}^2+B_{mn_2}^2)\pm\left(\frac{2c}{\hbar}\right)A_{mn_1}B_{mn_2}\\ &\times\left[\left(\frac{Mc}{2\hbar}\right)^2+\frac{M}{2}(A_{mn_1}^2-B_{mn_2}^2)(a_1-a_2)\right]^{1/2}\Bigg\}(A_{mn_1}^2-B_{mn_2}^2)^{-2}\end{aligned} \tag{3.23}$$

where

$$A_{mn_1}=\nu_{1m}+1+2n_1 \qquad (n_1=0,1,2,\ldots) \tag{3.24a}$$

and

$$B_{mn_2}=\nu_{2m}+1+2n_2 \qquad (n_2=0,1,2,\ldots). \tag{3.24b}$$

Equation (3.23) for the spectrum is valid when $k_1\neq k_2$ and $A_{mn_1}^2\neq B_{mn_2}^2$ (or $A_{mn_1}\neq B_{mn_2}$, since $A_{mn_1}, B_{mn_2}>0$). When $A_{mn_1}=B_{mn_2}$ this implies that

$$\nu_{1m}-\nu_{2m}=2n \qquad (n=0,\pm1,\pm2,\ldots) \tag{3.25}$$

so the energy levels are given by

$$E_{mn}=\frac{A_{mn_1}}{2c^2}\frac{\hbar^2}{M}(a_1-a_2)^2+\tfrac{1}{2}(a_1+a_2)+\frac{c^2M}{8\hbar^2A_{mn_1}^2}. \tag{3.26}$$

Note that equations (3.23) and (3.26) are both complementary to each other since they describe different parts of the same spectral problem for $A_{mn_1}\neq B_{mn_2}$ and $A_{mn_1}=B_{mn_2}$, respectively.

In the more restricted case where $a_1=a_2$, which implies $k_1=k_2$, the spectrum is given by

$$E_{mn_1n_2}=a_1-\frac{c^2M}{2\hbar^2(A_{mn_1}+B_{mn_2})^2} \tag{3.27}$$

which is the spectrum of the Hartman potential [7] and the Coulomb plus an Aharonov–Bohm potential [9].

4. Parabolic cylindrical coordinates

Following the same steps of the preceding section but now working with parabolic cylindrical coordinates u, v and z defined by

$$x=u^2-v^2 \qquad y=2uv \qquad z=z \tag{4.1}$$

where

$$-\infty < u < +\infty \qquad 0 \leqslant v < +\infty \qquad -\infty < z < +\infty$$

we propose the potential

$$V(u, v, z) = [4(u^2+v^2)]^{-1} \times [-4\alpha + (\beta-\gamma)u^{-2} + (\beta-\gamma)v^{-2} + 4\delta u^2 + 4\varepsilon v^2] + \alpha' z^2 + \beta' z^{-2} \qquad (4.2)$$

which is described by the SO(2, 1)⊗SO(2, 1)⊗SO(2, 1) dynamical group. This triple product is due to the fact that all the three coordinates u, v and z are associated with potentials, each one described by the SO(2, 1) group. As the solution for the potential $V(u, v, z)$ is quite similar to that of the $V(\xi, \eta)$ (equation (3.3)) we will only quote the energy spectrum of potential (4.2), which is given by

$$4\alpha = \lambda_1(\mu_1 + 1 + 2n_1) + \lambda_2(\mu_2 + 1 + 2n_2) \qquad (n_{1,2} = 0, 1, 2, \ldots) \qquad (4.3)$$

where

$$\lambda_1 = 2\hbar[2(E_m - \delta)/M]^{1/2} \qquad \lambda_2 = 2\hbar[2(E_m - \varepsilon)/M]^{1/2}$$
$$\mu_1 = [1 + 8M(\beta-\gamma)/\hbar^2]^{1/2}/2 \qquad \mu_2 = [1 + 8M(\beta+\gamma)/\hbar^2]^{1/2}/2 \qquad (4.4)$$
$$E_m = E - \hbar(2\alpha'/M)^{1/2}(\nu + 1 + 2m) \qquad \nu = [1 + 8M\beta'/\hbar^2]^{1/2}/2.$$

The explicit expressions for the Green function, wavefunctions and energies E can be obtained as in the case of parabolic rotational coordinates. When we put $\delta = \varepsilon = 0$ in potential (4.2) these results immediately reproduce the particular ones known in the literature [15, 22].

5. Conclusions

In this work we have shown that we can construct general potentials in parabolic coordinate systems which are exactly solvable since they are described by direct products of SO(2, 1) Lie groups. Many particular cases of these potentials [9, 15, 22] has been discussed in the literature.

It is the hope of the authors that this discussion can be extended to arbitrary three-dimensional systems and also to include relativistic quantum problems.

Acknowledgments

The authors would like to acknowledge the partial financial support given by Conselho Nacional de Desenvolvimento Científico e Tecnológico and Financiadora de Estudos e Projetos (Brazilian agencies).

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